MATH-SHU 350

Recitation 5: Inequalities and L^p Sapce

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Exercise 1 (Generalized Hölder's inequality)**.** *Assume that* $p \in (0,\infty]$ *and* $p_1, p_2 \cdots, p_n \in (0,\infty]$ *such that*

$$\frac{1}{p} = \sum_{k=1}^{n} \frac{1}{p_k}.$$

Then for any measurable function $\{f_k\}_{1\leqslant k\leqslant n}$*, we have*

$$\left\| \prod_{k=1}^{n} f_k \right\|_{L^p} \leqslant \prod_{k=1}^{n} \|f_k\|_{L^{p_k}}.$$

Exercise 2. *Let* $x_1 \cdots x_n$ *be positive numbers such that* $\sum_{i=1}^n x_i = 1$*. Prove that*

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1-x_i}} \geqslant \sqrt{\frac{n}{n-1}}.$$

Exercise 3. *Let* $0 \leqslant X_1 \leqslant X_2 \cdots$ *be random variables with* $\mathbb{E}[X_n] \sim an^\alpha$ *with* $\alpha > 0$*, and* $\mathrm{Var}[X_n] \leqslant Bn^\beta$ *with* $\beta < 2\alpha$*. Show that* $X_n/n^\alpha \to a$ *a.s.*

Exercise 4. *let* X_n *be independent Poisson random variables with* $\mathbb{E}[X_n] = \lambda_n$*, and let* $S_n = \sum_{i=1}^n X_i$*. Show that if* $\sum_n \lambda_n = \infty$*, then* $S_n/\mathbb{E}[S_n] \to 1$ *a.s.*

Exercise 5 (Shannon entropy)**.** *Suppose that* X *is a random variable taking values* x_i *with probability* p_i*, with* $0 < p_i < 1$*. Define Shannon entropy by*

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i.$$

Show that $H(X) \leqslant \log n$ *with equality if and only if* $p_i = 1/n$ *for all* i*.*

Exercise 6 (Interpolation inequality)**.**

1. *Prove that if random variable* $X \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ *for some* $p > 1$*, then for any* $p' \in [1,p]$*,* $X \in L^{p'}(\Omega, \mathcal{F}, \mathbb{P})$*.*
2. *Show that the same statement does not hold for* L^p *space on* $\mathbb{R}^d$ *with respect to the Lebesgue measure. More precisely, give some counter example that for any* $1 \leqslant p' < p$*,* $L^p(\mathbb{R}^d) \not\subset L^{p'}(\mathbb{R}^d)$*, and* $L^{p'}(\mathbb{R}^d) \not\subset L^p(\mathbb{R}^d)$*.*
3. † *Prove that if* $1 \leqslant p \leqslant r \leqslant q$*, then if* $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$*, then* $f \in L^r(\mathbb{R}^d)$*.*